A Course in Fluid Mechanics

CONTINUITY

MODULE

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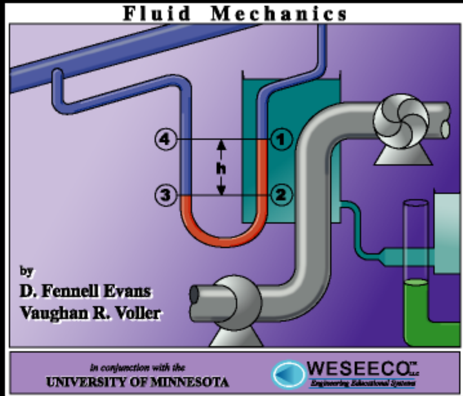

NAVIGATION

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TOC

The goals for this module are to

- Develop the concepts of systems and control volumes;
- Introduce continuity equations for both volume and mass flow rates.

Mass balances provide a useful starting point for analyzing flow in fluid systems.

The law of conservation of mass states that mass cannot be destroyed or created.

input = output + accumulation

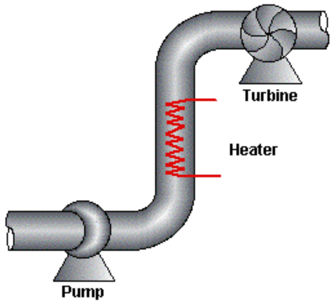
An equation that expresses conservation of mass is referred to as a continuity equation.

In many cases, there is no accumulation of material in the system

input = output

That is, what goes in must come out.

When a system's properties do not change with time, we say that it is in steady state.



In analyzing a complex fluid process, we often need to focus on a single feature.

For example, if we wanted to determine the flow rate, we could fix our attention on the fluid entering and leaving the pipe.

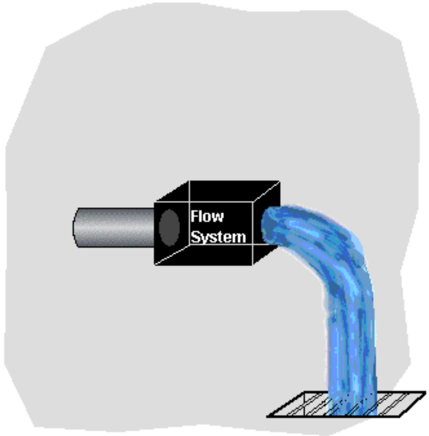
Here are some useful rules for carrying out analyses of this kind:

(1) Establish an imaginary boundary that isolates the feature of interest. In this case, we draw a box that cuts across the ends of the pipe.

(2) Define the surroundings as everything outside the boundary.

(3) Define the flow system as everything inside the boundary.

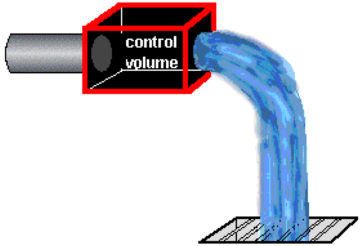
(4) Determine the interactions between the flow system and the surroundings.



To perform a mass balance, we need only know the flow conditions entering and leaving our system.

We construct a control surface that coincides with the flow system boundary; the control surface encloses the control volume.

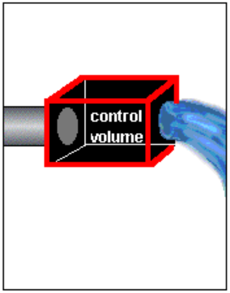
Control volumes are commonly used because:



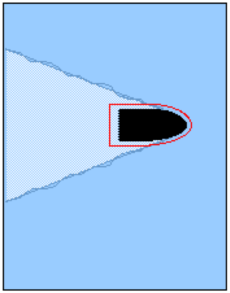
- We do not need to know conditions away from or inside the control volume but only at the entrance and exit.
- Changes even in complicated flow systems are easily accounted for by entrance and exit conditions.
- They do not need to be constant in size, shape, or fixed in space; they can be deformable or moving.
- For convenience, we can construct whatever control volume we wish; if properly analyzed, the results will always be the same.

So we can define the control volume in the way that provides the simplest analysis for a specific problem

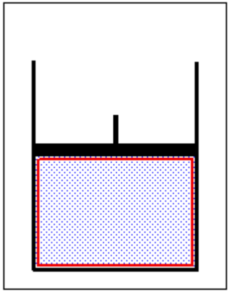
Some examples:



stationary
control volume



moving
control volume
(boat)



deformable
control volume
(piston and cylinder)

Volume Flow Rate

The goals of this section are to

- Introduce the volume flow rate, Q, and the average fluid velocity, v;
- Develop the continuity equation for steady-state flows;
- Examine the continuity equation for a system with multiple inputs and/or outputs.

To perform a mass balance, we need to know the volume flow rate, Q.

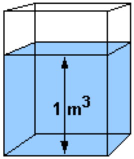
A simple experiment will determine Q for water flowing through the pipe.

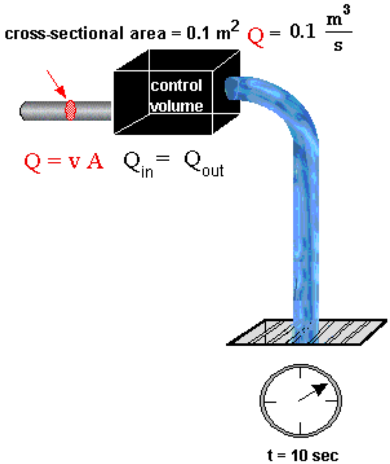
We can measure Q by finding out how long it takes for $1\ m^3$ of water to flow into a tank.

$$Q \text{ (volumetric flow rate)} = \frac{V \text{ (volume of water)}}{t \text{ (time)}}$$

$$= \frac{1\ m^3}{10\ s}$$

$$= 0.1\ \frac{m^3}{s}$$



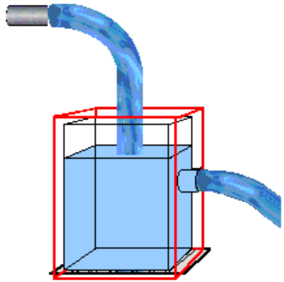


Now from the definition Q = vA

we can calculate the average fluid velocity:

$$v = \frac{Q}{A} = \frac{0.1\ \frac{m^3}{s}}{0.1\ m^2} = 1\ \frac{m}{s}$$

Let's modify our experiment by adding a drainage pipe to the tank to look at the conservation of mass as the tank is filled.



$Q_{in} = Q_{out} + \text{accumulation}$

We define a new control surface and analyze what happens in the new control volume consisting of the tank.

At the beginning of the experiment, when filling starts

$Q_{in} = \text{accumulation}$ (since no water is flowing out of the control volume)

Once the water level reaches the drainage pipe

$Q_{in} = Q_{out} + \text{accumulation}$ (water is still accumulating)

When the fluid level in the tank remains constant, there is no more accumulation and steady state is achieved.

$Q_{in} = Q_{out}$

In steady state flow with one input and one output

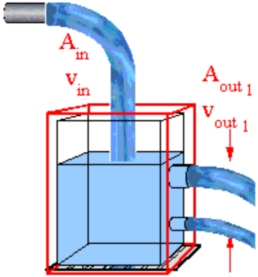
$$Q_{in} = Q_{out}$$

If we add a second drainage pipe, the water level will drop.

When a new steady state is reached

$$Q_{in} = Q_{out_1} + Q_{out\ 2}$$

or $$A_{in}\ v_{in} = A_{out\ 1}\ v_{out\ 1} + A_{out\ 2}\ v_{out\ 2}$$



It is often convenient to simplify this continuity equation by using summation, Σ, symbols.

$$\sum A\ v\,|_{in} = \sum A\ v\,|_{out}$$

Mass Flow Rate

The goals of this section are to

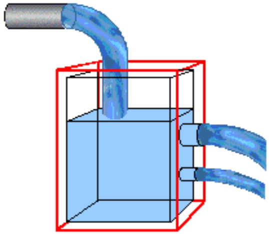
- Introduce the mass flow rate, $\dot{m}$;
- Show how the continuity equation is expressed in terms of mass flow rates;
- Examine the continuity equation for unsteady flow.

The volume form of the continuity equation is easy to analyze. However, in applications where density is not constant, volume varies as well, and the mass flow rate, $\dot{m}$, must be used.

$$\dot{m} = \rho Q = \rho v A \qquad \text{[kg/s]}$$

The continuity equation expressed in terms of mass flow rate is

$$\text{rate of change of mass} = \dot{m}_{in} - \dot{m}_{out}$$
$$\text{mass} = \rho_{in} Q_{in} - \rho_{out} Q_{out}$$
$$= \Sigma \rho A v\big|_{in} - \Sigma \rho A v\big|_{out}$$



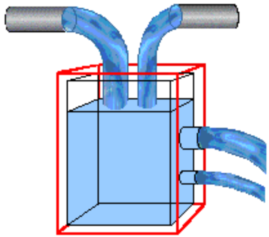
In steady state, the rate of change of mass is zero

$$\Sigma \rho A v\big|_{in} = \Sigma \rho A v\big|_{out}$$

So far, we assumed that the amount (mass and volume) of water in the tank remains constant, once steady state is achieved.

Then $\dot{m}_{in} = \dot{m}_{out}$

If we now add a second feed stream to the control volume, such that there is more inflow than outflow, the tank will begin to fill.



$$\text{rate of change of mass} = \dot{m}_{in} - \dot{m}_{out}$$

$$= \Sigma \rho A v|_{in} - \Sigma \rho A v|_{out}$$

The rate of change of mass is equivalent to the change in ρ V with time.

$$\frac{d(\rho V)}{dt} = \rho \frac{dV}{dt} \quad \text{assuming constant density}$$

Thus, in a system containing an incompressible fluid, mass accumulation results in a change in volume.

$$\rho \frac{dV}{dt} = \Sigma \rho A v|_{in} - \Sigma \rho A v|_{out}$$

For incompressible flow (ρ is constant) , the continuity equation

$$\rho \frac{dV}{dt} = \Sigma \rho_{in} Q_{in} - \Sigma \rho_{out} Q_{out}$$

becomes

$$\frac{dV}{dt} = \Sigma\ Q_{in} - \Sigma\ Q_{out}$$

$$= Q_1 + Q_2 - Q_3 - Q_4$$

$$= Q_1 + A_2 v_2 - A_3 v_3 - Q_4$$

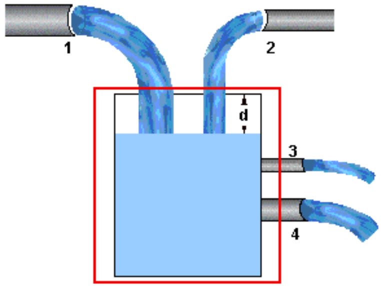
$$= [0.1 + (0.05)(1.5) - (0.015)(1.5) - 0.15]\ \frac{m^3}{s}$$

$$= +\ 0.0025\ \frac{m^3}{s}$$

If the area of the free surface of the tank is 1 m^2, then the water will rise at a rate of 0.0025 m/s. We can then find how long it will take the tank to overflow.

$$t_{overflow} = \frac{d}{v_{rise}} = \frac{0.45\ m}{0.0025\ \frac{m}{s}} = 180\ s = 3\ min$$

Given	
$A_1 = 0.1\ m^2$	$v_2 = v_3 = 1.5\ \frac{m}{s}$
$A_2 = 0.05\ m^2$	$Q_1 = 0.1\ \frac{m^3}{s}$
$A_3 = 0.015\ m^2$	$Q_4 = 0.15\ \frac{m^3}{s}$
$A_4 = 0.025\ m^2$	$d = 0.45\ m$



For incompressible flow (ρ is constant), the continuity equation

$$\rho\frac{dV}{dt} = \Sigma \rho_{in} Q_{in} - \Sigma \rho_{out} Q_{out}$$

becomes

$$\frac{dV}{dt} = \Sigma Q_{in} - \Sigma Q_{out}$$

$$= Q_1 + Q_2 - Q_3 - Q_4$$

$$= Q_1 + A_2 v_2 - A_3 v_3 - Q_4$$

$$= [0.1 + (0.05)(1.5) - (0.015)(1.5) - 0.15]\ \frac{m^3}{s}$$

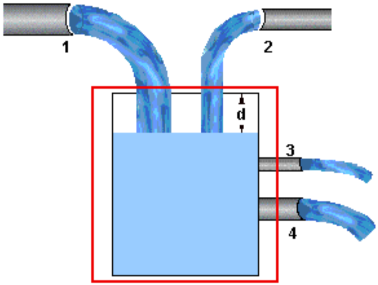
$$= +\ 0.0025\ \frac{m^3}{s}$$

If the area of the free surface of the tank is 1 m^2, then the water will rise at a rate of 0.0025 m/s. We can then find how long it will take the tank to overflow.

$$t_{overflow} = \frac{d}{v_{rise}} = \frac{0.45\ m}{0.0025\ \frac{m}{s}} = 180\ s = 3\ min$$

Given

$A_1 = 0.1\ m^2$	$v_2 = v_3 = 1.5\ \frac{m}{s}$
$A_2 = 0.05\ m^2$	$Q_1 = 0.1\ \frac{m^3}{s}$
$A_3 = 0.015\ m^2$	$Q_4 = 0.15\ \frac{m^3}{s}$
$A_4 = 0.025\ m^2$	$d = 0.45\ m$



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General Mass Balance

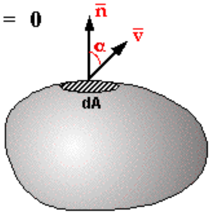
The goals of this section are to

- Generalize mass balances to any arbitrary, stationary control volume with fluid flowing through it;
- Identify conditions for maximum inflow, maximum outflow, and no flow across the control surface;
- Show the velocity distributions for laminar, turbulent, and plug flow.

In the previous examples, we employed control volumes that were readily defined in terms of a pipe network or a tank. We can generalize mass balances to any arbitrarily-shaped, stationary control volume with fluid flowing through it. The control surface encloses the control volume.

$$\begin{pmatrix}\text{net rate of mass}\\ \text{from control volume}\end{pmatrix} + \begin{pmatrix}\text{rate of mass accumulation}\\ \text{in control volume}\end{pmatrix} = 0$$

$$\iint_A \rho(\bar{n}\cdot\bar{v})dA + \frac{\partial}{\partial t}\iiint_V \rho dV = 0$$



We focus our analysis on a differential area, dA, on the surface of the control volume.

By convention, an outward unit vector, $\bar{n}$, is defined, normal to this area.

We denote the direction of fluid flow by the vector, $\bar{v}$, of magnitude v.

Mass flow rate $= (\rho v)(\cos\alpha\, dA) = \rho(\bar{n}\cdot\bar{v})dA$

Integration over the control surface gives

rate of mass from control volume $= \iint_A \rho v \cos\alpha\, dA = \iint_A \rho(\bar{n}\cdot\bar{v})dA$

Rate of mass accumulation in the control volume

$$\frac{dm}{dt} = \frac{\partial}{\partial t}\iiint_V \rho dV$$

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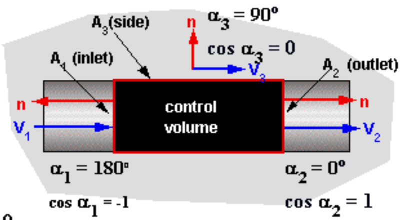
The general mass balance equation can then be written as

$$\iint_A \rho(\bar{n}\cdot\bar{v})dA + \frac{\partial}{\partial t}\iiint_V \rho dV = 0 \xrightarrow[\text{state}]{\text{steady}} \iint_A \rho(\bar{n}\cdot\bar{v})dA = 0$$

Applying this general equation to a one-dimensional, steady-state flow:



$$\iint_A \rho(\bar{n}\cdot\bar{v})dA = 0$$

Expanding as integrals over each area, A_1, A_2, and A_3 of the control volume.

$$\iint_A \rho(\bar{n}\cdot\bar{v})dA = \iint_A \rho v_1 \cos\alpha_1 dA + \iint_A \rho v_2 \cos\alpha_2 dA = 0$$

Upon integration, the right-hand side becomes

$$\iint_A \rho(\bar{n}\cdot\bar{v})dA = \rho v_2 A_2 - \rho v_1 A_1 = 0$$

The momentum flux arising from an incompressible fluid (ρ constant) flowing through a pipe of constant diameter ($A_1 = A_2$) is always zero

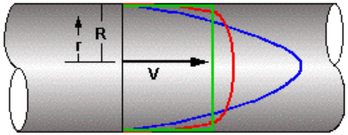
Up to this point, when calculating the integrals in the continuity equation, we have assumed constant velocities across the pipe entrance and exit. In reality, this may not be the case.

$$\iint_A \rho(\bar{n}\cdot\bar{v})dA = \iint_A \rho v_1 \cos\alpha_1 dA_1 + \iint_A \rho v_2 \cos\alpha_2 dA_2$$

In laminar flow the velocity distribution is parabolic.

$$v = v_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

$$v_{av} = \frac{1}{A}\iint_A v\,dA = \frac{v_{max}}{2}$$

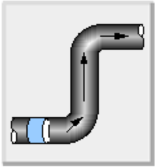


laminar flow profile
turbulent flow profile
plug flow profile

In fully turbulent flow the velocity distribution can be expressed as

$$v = v_{max}\left(\frac{R-r}{R}\right)^{\frac{1}{7}} \quad \text{and} \quad v_{av} \approx 0.8 v_{max}$$

We often approximate turbulent flow as plug flow, $v_{av} = v_{max}$.



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